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A Calculation of Radiation Effects on Electron Oscillations in a Circular Accelerator.

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1. – In the magnetic field of a cyclic accelerator the electrons oscillate around an equilibrium orbit.

Three modes of oscillations are possible; which, under suitable conditions, can be considered uncoupled: radial betatron oscillations, vertical betatron oscillations and synchrotron oscillations.

The radiofrequency accelerating fields and the radiation losses damp down each oscillation mode. These damping rates have been evaluated by several authors.

SOKOLOV and TERNOV (¹) analysed the problem determining the quantum states of an electron in the magnetic field of the accelerator and treating the radiation as a perturbation coupling these states.

ROBINSON (²) and KOLOMENSKII and LEBEDEV (³) considered the damping as produced by the classical self-force due to radiation; further they also took into account the random character of photon emission which induces oscillations.

These have been the main lines of approach to the problem.

While the results obtained for the values of the damping rates are all in agreement for the case of a constant gradient machine, it is not so for an accelerator with a more complicated structure.

There is now in many laboratories a great interest in storage rings requiring quite complicated magnet systems.

(¹) A. A. SOKOLOV and I. M. TERNOV: *Sov. Phys. JETP*, **1**, 277 (1955).
(²) K. W. ROBINSON: *Phys. Rev.*, **111**, 373 (1958).
(³) A. A. KOLOMENSKII and A. N. LEBEDEV: *CERN Symposium* (1956), p. 447.

It is necessary in these machines to have damping on all the three modes of oscillations. The possible means suggested by ROBINSON and ORLOV, TARASOV, KHEJFETS [4] or by KOLOMENSKII and LEBEDEV to reach such a situation differ strongly.

Following ROBINSON what matters is the average energy radiated per turn which depends on the whole magnet structure.

In the other case (KOLOMENSKII and LEBEDEV), the important element is the average of the local gradient of the magnetic field. All this makes clear that it is desirable to have a new independent evaluation of the damping rates and of the equilibrium dimensions of the beams. In fact these are an important element for a storage ring, since they contribute to determine the number of reactions which take place in the machine.

In what follows we shall first illustrate our approach to the problem by means of a simple example; afterwards the real case will be studied.

2. – Let us consider a harmonic oscillator colliding with particles of an external medium so that it exchanges momentum and energy with them at random and instantaneously.

We can think of this as of a model of a particle in brownian motion and subject to an elastic force.

We will assume that the spectrum of the energy given to the oscillator in one collision is defined and known and that the mean value of the energy exchanged per period is zero.

In the absence of collisions the motion is described by the equation

$$\ddot{x} + \omega^2 x = 0 \tag{1}$$

with the solution

$$x = x_0 \cos(\omega t + \varphi) . \tag{2}$$

What will be the motion when the interaction is switched on?

Let us suppose that three collisions take place at the times t_0, t_1, t_2, and no one during the intervals $t_0 - t_1$, $t_1 - t_2$.

In these intervals the motion is still described by (1) and if the equation is solved the solutions in $t_0 - t_1$, $t_1 - t_2$ can only differ by the values of x_0 and φ.

The variation of x_0 due to one collision can be easily evaluated.

As a matter of fact (1) has a first integral

$$x_0^2 = x^2 + \frac{\dot{x}^2}{\omega^2} . \tag{3}$$

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[4] Y. F. ORLOV, E. K. TARASOV and S. A. KHEIFETS: *International Conference on High-Energy Accelerators CERN* (1958), p. 306.

If a momentum q is exchanged at the time t_i the oscillator velocity becomes $\dot{x}(t_i) + \delta\dot{x}$ and

$$\delta\dot{x} = -\frac{q}{m}. \tag{4}$$

From (3) it follows that

$$\overline{x}_0^2 = x^2 + \frac{(\dot{x}+\delta\dot{x})^2}{\omega^2} = x_0^2 + \left\{-\frac{2\dot{x}q}{\omega^2 m} + \frac{q^2}{\omega^2 m^2}\right\}.$$

The rate of change of x_0^2 is then

$$\Delta(x_0^2) = \left\{-2\frac{\dot{x}q}{\omega^2 m} + \frac{q^2}{m^2\omega^2}\right\}\delta(t-t_i). \tag{5}$$

Now let us assume that the momentum spectrum $p(q)$ is such that the average values of q and q^2 are

$$\int_0^\infty qp(q)\,\mathrm{d}q = M\dot{x},$$

$$\int_0^\infty q^2 p(q)\,\mathrm{d}q = \lambda.$$

Substituting in (5) q and q^2 with $M\dot{x}$ and λ and integrating over one period we get $D(x_0^2)$, the average variation of x_0^2 per period,

$$D(x_0^2) \equiv \frac{1}{T}\int_0^T \Delta(x_0^2)\,\mathrm{d}t = -\frac{M\eta}{m}x_0^2 + \frac{\lambda\eta}{\omega^2 m^2}. \tag{6}$$

In deriving (6) the number η of collisions per period is supposed to be very large and further $M\eta/m \ll 1$ is assumed.

From (6) it is possible to get the value of x_0^2 after n periods:

$$x_0^2(n) = \left(x_0^2(0) - \frac{\lambda}{\omega^2 mM}\right)\exp\left[-\frac{M}{m}\eta nT\right] + \frac{\lambda}{\omega^2 Mm}. \tag{7}$$

It is seen that a large number of collisions gives rise to a damping of the oscillations until their amplitude is reduced to a value independent of the initial conditions.

In the next sections the equations of motion of the electrons in a circular accelerator will be shortly derived and afterwards the above considerations will be extended to this case.

3. – Neglecting radiation the motion of the electrons is described by the classical equations

$$m_0 c \frac{\mathrm{d}U_i}{\mathrm{d}t} = \frac{e}{c} F_{ik} U_k \,, \tag{8}$$

where U_i is the electron four-velocity and F_{ik} is in our case the guide magnetic field.

Let us introduce a reference trajectory (R.T.), which can be thought of as the trajectory of an ideal electron of energy E_s moving in the accelerator without gaining or losing energy. The position of any electron will be referred to R.T. in the following way (*):

Let s be the arc length on R.T.; $P(s)$ one of its points; $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ an orthonormal triad such that, if $\mathrm{d}f/\mathrm{d}s = f'$,

$$\begin{cases} \boldsymbol{\lambda}_1 = \dfrac{\mathrm{d}P}{\mathrm{d}s} \,, \\ K\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1' \,, \\ -K\boldsymbol{\lambda}_1 + H\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_2' \,, \\ -H\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_3' \,, \end{cases} \tag{9}$$

with K and H curvature and torsion of R.T.; then the position of an electron will be $P + \delta P$ where

$$\begin{cases} \delta P(s) = \boldsymbol{\lambda}_1 \sigma(s) + \boldsymbol{\lambda}_2 x(s) + \boldsymbol{\lambda}_3 z(s) \,, \\ \sigma(s = 0) = 0 \,. \end{cases} \tag{10}$$

In the following R.T. will be assumed to be a plane curve ($H = 0$); F_{ik} to be a static magnetic field $\mathscr{H}$. The R.F. will simply compensate the radiation losses.

Every vector will be decomposed along the $\boldsymbol{\lambda}_i$:

$$\boldsymbol{v} = v_i \boldsymbol{\lambda}_i \qquad (i = 1, 2, 3).$$

The value that a physical quantity assumes on R.T. will be labelled by a subscript « s »

δP is assumed to be small of the first order so that quantities like $\mathscr{H}$ can be expanded as a power series in σ, x, z neglecting second order terms.

(*) The use of this frame of reference was suggested to me by C. BERNARDINI.

Other notations introduced are

$$n = \frac{1}{K\mathscr{H}_{3s}} \frac{\partial \mathscr{H}_3}{\partial x}, \tag{11}$$

$$p = \frac{E - E_s}{E_s}, \tag{12}$$

where E is the electron energy.

Then $\mathscr{H}$ is given by

$$\begin{cases} \mathscr{H}_1 \equiv 0 , \\ \mathscr{H}_2 = \mathscr{H}_{3s} K n z , \\ \mathscr{H}_3 = \mathscr{H}_{3s}(1 + Knz) , \\ |\mathscr{H}_{3s}| = \dfrac{K v_s}{ec} E_s , \qquad \mathscr{H}_{3s} < 0 . \end{cases} \tag{13}$$

The linearized equations of motion are

$$\begin{cases} \sigma' = Kx , \\ x'' + K^2(1 - n)x = - Kp , \\ z'' + K^2 n z = 0 , \\ p = \text{cost} . \end{cases} \tag{14}$$

In writing down (14) terms in $\delta v/v = (v - v_s)/v_s$ have been neglected. This is justified because at the energy considered

$$\frac{\delta v}{v} = p \frac{m_0^2 c^6}{E_s^2 v^2} \ll p ,$$

and p is assumed to be of the same order of Kx, Kz.

4. – The solution of the equations (14) for x and z have been widely discussed for the case in which K and n are periodic functions of s [5].

Using the same notations as in reference [5] the solution for the x mode

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[5] F. D. COURANT and H. S. SNYDER: *Ann. of Phys.*, **3**, 1 (1958).

can be written

$$x = x_0\sqrt{\beta_r}\cos(\nu_r\varphi_r + \xi_r) - p\psi\,, \tag{15}$$

$$\psi = \nu_r\sqrt{\beta_r}\sum_0^\infty {}_l\,\frac{a_l}{\nu_r^2 - l^2}\exp[il\varphi_r]\,, \tag{15'}$$

$$a_l = \frac{1}{2\pi\nu_r}\int\limits_0^{C_s} l\sqrt{\beta_r}\exp[-il\varphi_r]\,\mathrm{d}s\,. \tag{15''}$$

C_s is the length of R.T. and if L is the arc length for one period of the machine then β_r has the period L.

$$x_\beta = x_0\sqrt{\beta_r}\cos(\nu_r\varphi_r + \xi_r) = x_0\sqrt{\beta_r}\cos\gamma_r\,, \tag{16}$$

describes the radial betatron oscillations.

The term $-p\psi$ gives the deviations from R.T. due to the energy difference $E - E_s$.

The solution for the vertical mode is

$$z = z_0\sqrt{\beta_v}\cos(\nu_v\varphi_v + \xi_v) = z_0\sqrt{\beta_v}\cos\gamma_v\,, \tag{17}$$

in complete analogy with (16).

5. – (15), (17) describe the motion of one electron with respect to R.T. when the radiation and the accelerating cavities are neglected.

When these are included the synchrotron mode of oscillations, *i.e.* an oscillation for p, comes in. To derive an equation for the oscillations we assume that one accelerating cavity is present in the point $s = lC_s$ (l is an integer number) and that it furnishes to the electrons an energy

$$\varepsilon_r = \frac{eV_0\cos\varphi}{E_s} = \frac{eV_0}{E_s}\cos\varphi_s + f^2\sigma(lC_s)\,, \tag{18}$$

where φ_s is the synchrotron phase and

$$f^2 = \frac{2\pi KeV_0}{C_sE_s}\sin\varphi_s\,. \tag{19}$$

Further the energy spectrum of the radiated photons is supposed to be the classical one, so that if ε is the energy of a single photon (divided by E_s), the average values of ε and ε^2 are given by ([6,7])

$$\langle\varepsilon\rangle = w_s(1 + 2nKx + 2p) = w \tag{20}$$

$$\langle\varepsilon^2\rangle = \frac{55}{24\sqrt{3}}\, r_e \Lambda \gamma^5 K^3 , \tag{21}$$

where

$$w_s = \frac{2}{3}\, r_e \gamma^3 K^2 , \qquad \gamma = \frac{E_s}{m_0 c^2} , \qquad r_e = \frac{e^2}{m_0 c^2} , \qquad \Lambda = \frac{\hbar}{m_0 c} .$$

$E_s\langle\varepsilon\rangle$ is the radiated energy per unit arc length; (20) can be obtained from the fourth component of the self-force acting on the electron ([8]).

To assume that the photon energy spectrum is given by the classical one is correct in the energy region we are considering ([9]).

Considering only zero order terms of (20) and assuming

$$eV_0/E \cos\varphi_s = \langle w_s\rangle\, C_s \tag{22}$$

we get the variation of p per turn:

$$Dp = f^2 \frac{\sigma(lC_s)}{C_s} .$$

The second variation is then

$$D^2 p = \frac{f^2}{C_s}\, \frac{\sigma[(l+1)C_s] - \sigma(lC_s)}{C_s} ,$$

and since $\sigma' = Kx \simeq - Kp\psi$

$$D^2 p = - \nu_s^2 p , \tag{23}$$

where

$$\nu_s^2 = \frac{f^2 \alpha}{C_s} . \tag{24}$$

([6]) J. SCHWINGER: *Phys. Rev.*, **75**, 1912 (1949).

([7]) See, for example, M. SANDS: *Phys. Rev.*, **97**, 420 (1955).

([8]) L. D. LANDAU and E. M. LIFSHITZ: *The Classical Theory of Fields* (Londra, 1959), p. 233.

([9]) The limit of validity of this approximation is given, for example, by A. A. KOLOMENSKY and A. N. LEBEDEV: *CERN Symposium* (1956), p. 447.

and

$$\alpha = \frac{1}{C_s}\int_0^{C_s} K\psi \, ds \,, \tag{25}$$

is the momentum compaction factor.

The solution of (23) is

$$p = p_0 \cos(s\nu_s + \pi) \,. \tag{26}$$

6. – Let us consider an electron emitting or absorbing instantaneously a photon of momentum $\boldsymbol{q}$, energy cq; as a consequence its position, velocity and energy will change as follows:

$$\begin{cases} \delta x = \delta z = \delta\sigma = 0 \,, \\ \delta x' = -c\,\dfrac{\boldsymbol{q}\times\boldsymbol{\lambda}_2 + q(x' + K\sigma)}{E} \,, & \delta\sigma' = 0 \,, \\ \delta z' = -c\,\dfrac{q\times\boldsymbol{\lambda}_3 + qz'}{E} \,, & \delta p = \dfrac{cq}{E_s} = \varepsilon \,, \end{cases} \tag{27}$$

$\varepsilon > 0$ for an absorbed photon, $\varepsilon < 0$ for an emitted one.

The variations in x', y', σ' are obtained from

$$\delta\left(\frac{E}{c^2}\boldsymbol{v}\right) = -\boldsymbol{q} \,,$$

$$\boldsymbol{v} = \boldsymbol{v}_s + c\boldsymbol{\lambda}_2(x' + K\sigma) + c\boldsymbol{\lambda}_3 z' \,.$$

To see the effect on x_0, z_0, p_0 we express these quantities as functions of x, x', z, z'; *i.e.* we write the first integrals of (14), (23):

$$x_0^2 = \left\{(x + p\psi)^2 + \left[\frac{1}{2}\beta_r'(x + p\psi) - \beta_r(x' + p\psi')\right]^2\right\}\frac{1}{\beta_r} \,, \tag{28}$$

$$z_0^2 = \left\{z^2 + \left(\frac{1}{2}\beta_v' z - \beta_v z'\right)^2\right\}\frac{1}{\beta_v} \,, \tag{29}$$

$$p_0^2 = p^2 + \frac{p'^2}{\nu_s^2} \,. \tag{30}$$

Using (15), (17), (26) and substituting (27) in (28), (29), (30), $\overline{p}_0$, $\overline{x}_0$ and $\overline{z}_0$

are obtained:

$$(31)\quad \begin{cases} \overline{x}_0^2 = x_0^2 + 2x_0\,\delta x_0 + \delta x_0^2\,, \\ \delta x_0 = \dfrac{\varepsilon}{\sqrt{\beta_r}}\left\{\psi\cos\gamma_r + \left(\dfrac{1}{2}\psi\beta_r' - \beta_r\psi'\right)\sin\gamma_r\right\} - \delta x'\sqrt{\beta_r}\sin\gamma_r\,, \\ \delta x_0^2 = \dfrac{\varepsilon^2}{\beta_r}\left\{\psi^2 + \left(\dfrac{1}{2}\,\beta_r'\psi - \beta_r'\psi\right)^2\right\} + \beta_r\,\delta x'^2 - 2\varepsilon\,\delta x'\left(\dfrac{1}{2}\beta'\psi - \beta\psi'\right), \end{cases}$$

$$(32)\quad \begin{aligned} \overline{z}_0^2 &= z_0^2 + 2z_0\,\delta z_0 + \delta z\,, \\ \delta z_0 &= -\,\delta z'\sqrt{\beta_v}\sin\gamma_v\,, \end{aligned} \qquad \delta z_0^2 = \beta_v\,\delta z'^2\,,$$

$$(33)\quad \overline{p}_0^2 = p_0^2 + 2\varepsilon p_0\cos(s\nu_s + \pi) + \varepsilon^2\,.$$

Now consider the case in which a photon of energy ε is radiated at $s = s_i$; this formally corresponds to a rate of change of the energy $\varepsilon_i\delta(s - s_i)$. The direction of these photons makes with the electron velocity an angle Φ of order m_0c^2/E. This angle can be neglected, because the photons that matter are very soft, in the case of the radial betatron mode.

It follows from (11) that the corresponding variation in x', σ' is zero (to the order m_0c^2/E). Further we assume that only one R.F. cavity is present in the machine in the position $s = lC_s$ and it exchanges with the electron a photon of energy ε_r directed as $\boldsymbol{\lambda}_1$.

The rates of variation of p, x', are then

$$(34)\quad \begin{cases} \Delta p = \varepsilon_r\delta(s + \sigma - lC_s) - \varepsilon_i\,\delta(s - s_i)\,, \\ \Delta x' = -\,\varepsilon_r(x' + K\sigma)\,\delta(s + \sigma - lC_s)\,, \end{cases}$$

where l is an integer number.

In the case of the vertical oscillations it is not possible to neglect the angle Φ since it will be seen below that this is the only radiation term that contributes to the vertical dimensions of the beam.

Now the contribution to $\Delta z'$ of the R.F. is $-\varepsilon_r z'\,\delta(s+\sigma-lC_s)$.

Since the radiated photons are emitted in a cone making the angle Φ with $\boldsymbol{v}$, it is clear that they do not contribute in the average to $\Delta z'$ but only to $(\Delta z')^2$, so that

$$(35)\quad \begin{cases} \Delta z' = -\,\varepsilon_r z'\,\delta(s + \sigma - lC_s)\,, \\ (\Delta z')^2 = \varepsilon_r^2 z'^2\,\delta(s + \sigma - lC_s) + \overline{\Phi}^2\varepsilon_i^2\,\delta(s - s_i)\,. \end{cases}$$

7. – Let us derive now the equations for the amplitudes beginning with the vertical oscillations. From (32), (35) it follows that

$$(36)\quad \Delta(z_0^2) = +\,2z_0\varepsilon_r z'\,\delta(s + \sigma - lC_s)\sqrt{\beta_v}\sin\gamma_v + \\ +\,\overline{\Phi}^2\varepsilon_i^2\beta_v\,\delta(s - s_i) + \beta_v\varepsilon_r^2 z'^2\,\delta(s + \sigma - lC_s)\,.$$

Averaging (36) over N periods of oscillation we get (see Appendix)

$$D(z_0^2) = \frac{2}{\tau_{\beta_v}} z_0^2 + \frac{55}{24\sqrt{3}} r_e \Lambda \gamma^5 \Phi^2 \langle \beta_v K^3 \rangle , \tag{37}$$

where

$$\frac{1}{\tau_{\beta_v}} = + \frac{1}{4}\left\{1 + \frac{\beta_v'^2}{4}\right\}\bigg|_{s=C_s} C_s \langle w_s \rangle^2 - \frac{1}{2}\langle w_s \rangle \simeq -\frac{1}{2}\langle w_s \rangle , \tag{38}$$

is the damping rate (*).

In deriving (37) it has been assumed that the betatron frequency is not an integer multiple of $2\pi/C_s$ so that the phase γ_v takes on all the possible values in the point $s = C_s$.

To evaluate the damping for the radial betatron oscillations w is written in the form

$$w = w_s + \frac{\partial w}{\partial x_0} x_0 + \frac{\partial w}{\partial p} p = w_s + \frac{\partial w}{\partial x_\beta} x_\beta + \frac{\partial w}{\partial p} p .$$

Then from (31) it follows

$$\langle \Delta x_0 \rangle = -\frac{1}{NC} \sum_i \frac{\Delta s}{\sqrt{\beta_r}} \left[w_s + \frac{\partial w}{\partial x_\beta} x_\beta + \frac{\partial w}{\partial p} p \right]_{s=s_i} \cdot$$

$$\cdot \left[\psi \cos\gamma_r + \left(\frac{1}{2}\beta_r' \psi - \beta_r \psi'\right) \sin\gamma_r \right]_{s=s_i} +$$

$$+ \frac{1}{NC} \sum_h \langle w_s \rangle C_s \left\{ \frac{1}{\sqrt{\beta_r}} \psi \cos\gamma_r + \frac{1}{\sqrt{\beta_r}} \left(\frac{1}{2}\beta' \psi - \beta \psi'\right) \sin\gamma_r + \right.$$

$$\left. + x_0 \sqrt{\beta_r} \sin\gamma_r \left[\frac{1}{2} \frac{\beta_r'}{\sqrt{\beta_r}} \cos\gamma_r - \frac{1}{\sqrt{\beta_r}} \sin\gamma_r - p\psi' + K\sigma \right] \right\}_{s=hC_s} .$$

Averaging is performed on a number N of turns such that many betatron oscillations occur while the synchrotron oscillations can be neglected. This is possible because $\nu_s \ll \nu_r$. Taking into account the periodicity of the various terms one gets the result

$$\frac{1}{\tau_{\beta_r}} = -\frac{1}{2}\left\langle \frac{\partial w}{\partial x_{\beta_r}} \psi \right\rangle - \frac{1}{2}\langle w_s \rangle . \tag{39}$$

Also in this case the term $\beta_r \Delta x'^2$ in (31) gives a negligible contribution to $1/\tau_{\beta_r}$.

(*) The term $+\frac{1}{4}\{1 + \beta_v'^2/4\}|_{s=C_s} \langle w_s \rangle^2$, which derives from $\beta_v \Delta z'^2$ of (32), can be obviously neglected.

The evaluation of $\langle\Delta x_0^2\rangle$ leads to

$$\langle\Delta x_0^2\rangle = \frac{55}{24\sqrt{3}}\, r_e \Lambda\gamma^5 F_1 + \langle w_s^2\rangle F_2\,, \tag{40}$$

with

$$F_1 = \left\langle\left[\psi^2 + \left(\frac{1}{2}\beta_r'\psi - \beta_r\psi'\right)^2\right]\frac{K^3}{\beta_r}\right\rangle, \tag{41}$$

$$F_2 = \frac{1}{\beta_r}\left[\psi^2 + \left(\frac{1}{2}\beta_r'\psi - \beta_r\psi'\right)^2\right]_{s=0}. \tag{42}$$

In (40) F_1 represents the contribution due to radiation and F_2 the contribution due to R.F.

From (39), (40) the equation for the rate of change of the amplitude of the radial betatron oscillations is obtained:

$$D(x_0^2) = \frac{2x_0^2}{\tau_{\beta_r}} + \frac{55}{24\sqrt{3}}\, r_e\Lambda\gamma^5 F_1 + \langle w_s^2\rangle F_2\,. \tag{43}$$

8. – It remains to derive the equation for the amplitude of the synchrotron oscillations.

From (27) we get

$$\Delta p = \varepsilon_r\,\delta(s + \sigma - lC_s) - \varepsilon_i\,\delta(s - s_i)\,. \tag{44}$$

When (18), (20) are substituted in (44) it is necessary to remember that the terms ε_r and $\langle w_s\rangle$ were already taken into account in deriving (23), so that now

$$\Delta p = -\, w_s(2nKx + 2p)\,\delta(s - s_i)\,\Delta s\,. \tag{45}$$

Neglecting in x the betatron oscillation term, *i.e.* assuming $x = -p\psi$, eq. (33) becomes

$$\Delta(p_0^2) = \left\{-\,2p_0^2\cos^2(\nu_s s + \pi)w_s(2 - 2nK\psi) + \frac{55}{24\sqrt{3}}\, r_e\Lambda\gamma^5 K^3\right\}\Delta s\,\delta(s - s_i)\,.$$

Averaging we get

$$D(p_0^2) = \frac{2p_0^2}{\tau_s} + \frac{55}{24\sqrt{3}}\, r_e\Lambda\gamma^5\langle K^3\rangle\,, \tag{46}$$

where

$$\frac{1}{\tau_s} = -\,\frac{1}{2}\,\langle w_s(2 - 2nK\psi)\rangle\,. \tag{47}$$

9. – In this section we want to discuss the results obtained. Notice first that from the definition (20) it follows

$$\frac{\partial w}{\partial x_\beta}\psi = 2Knw_s\psi\,, \tag{48}$$

$$\frac{\mathrm{d}w}{\mathrm{d}p} = w_s(2 - 2nK\psi)\,. \tag{49}$$

Using (48), (49) the three damping constants (38), (39), (47) can be written as

$$\frac{1}{\tau_{\beta_v}} = -\frac{1}{2}\langle w_s\rangle\,, \tag{38}$$

$$\frac{1}{\tau_{\beta_r}} = \frac{1}{2}\left\langle\frac{\mathrm{d}w}{\mathrm{d}p}\right\rangle - \frac{3}{2}\langle w_s\rangle\,, \tag{39'}$$

$$\frac{1}{\tau_s} = -\frac{1}{2}\left\langle\frac{\mathrm{d}w}{\mathrm{d}p}\right\rangle. \tag{47'}$$

These results are the same as those given by ROBINSON [2].

When evaluating (39′) (47′) it is necessary to remember that the averages must be performed on the actual electron path C or, what is the same, on the R.T. provided that a term $+K\psi$ is added to (49).

From (37), (43), (46) we can get the root mean square equilibrium dimensions of the beams:

$$d_{\beta_v} = \left\{\frac{55\sqrt{3}}{24}\,\beta_{v\max}\Phi^2\Lambda\,\frac{\langle\beta_v K^3\rangle}{\langle K^2\rangle}\right\}^{\frac{1}{2}}\gamma\,, \tag{50}$$

$$d_{\beta_r} = \left\{\frac{55\sqrt{3}}{24}\,\beta_{r\max}\,\frac{\Lambda F_1}{\langle K^2(1+2nK\psi - K\psi)\rangle}\right\}^{\frac{1}{2}}\gamma\,, \tag{51}$$

$$d_s = \psi_{\max}\left\{\frac{55\sqrt{3}}{48}\,\frac{\Lambda\langle K^3\rangle}{\langle K^2(1-nK\psi)\rangle}\right\}^{\frac{1}{2}}\gamma\,. \tag{52}$$

(52) gives the dimensions of the beam due to the oscillation in the closed orbit $x = -p\psi$ associated with the oscillations in p. All the dimensions given by (50), (51), (52) are the maximum ones (along R.T.) as is shown by the factors $\beta_{v\max}$, $\beta_{r\max}$, $\psi_{\max}$.

In writing down (51) the term F_2 appearing in (43) was neglected.

The term F_1 in a first approximation valid for isomagnetic machines, *i.e.* assuming [5]

$$\psi \sim a_0\sqrt{\beta}\,, \qquad a_0 \sim \sqrt{\frac{R}{\nu^3}}\,,$$

$$\beta \sim \frac{R}{\nu}\,, \qquad \alpha \sim \frac{1}{\nu^2}\,,$$

R = average radius of the machine, becomes

$$F_1 \simeq \frac{\alpha^2 \nu^2}{R^2},$$

and this is in agreement with the formula given by ROBINSON [2].

The results (50), (51), (52) can be applied to a large class of circular accelerators, *i.e.* to all machines having the radial plane as plane of symmetry.

In particular they are valid for the strong focusing non isomagnetic machines which are now considered as possible storage devices, as they can have damping on all the three modes of oscillations.

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APPENDIX

In this Appendix we want to discuss in detail the derivation of equations (37), (43), (46).

The substituion of (18), (21) and (22) in (36) and the use of the relation

$$z' = \frac{z_0}{\sqrt{\beta_v}} \left\{ \frac{1}{2}\, \beta'_v \cos \gamma_v - \sin \gamma_v \right\}.$$

give us

$$\text{(A.1)}\quad \Delta(z_0^2) = 2z_0^2 \sin \gamma_v (\tfrac{1}{2}\beta'_v \cos \gamma_v - \sin \gamma_v)\, [\langle w_s \rangle C_s + f^1 \sigma(lC_s)] \cdot \delta(s + \sigma - lC_s) + \\ + z_0^2 (\tfrac{1}{2}\beta'_v \cos \gamma_v - \sin \gamma_v)^2 [\langle w_s \rangle C_s + f^2 \sigma(lC_s)]^2 \cdot \delta(s + \sigma - lC_s) + \\ + \frac{5}{24\sqrt{3}}\, r_e \Lambda \gamma^5 \overline{\Phi} \beta_v K^3 \, \Delta s \delta(s - s_i)\,.$$

Rememeber that the factor Δs, in the third term on the right-hand side, comes in because (20) and (21) represent the energy and the square of the energy radiated per unit arc length. In fact, we assume that the photon of energy ε_i has been radiated in the interval $s - s + \Delta s$, so that the total energy radiated in a turn is equal to $C\langle w \rangle$:

$$\text{(A.2)}\qquad \sum_1^M {}_i\, \varepsilon_i = \sum_1^M {}_i\, w(s_i) \Delta s = \int_0^C w(s)\, \mathrm{d}s = C\langle w \rangle\,.$$

Here M is the average number of photons emitted during one turn and it is assumed to be very large.

Going back to (A.1), and neglecting terms $f^2\sigma$, we get for the average variation of z_0^2 per turn

$$\text{(A.3)}\qquad D(z_0^2) = \frac{1}{NC}\int_0^{NC}\Delta(z_0^2)\,\mathrm{d}s =$$
$$= \frac{1}{NC}\sum_1^N{}_h\left\{2z_0^2\langle w_s\rangle C_s\left(\tfrac{1}{2}\beta_v'\cos\gamma_v\sin\gamma_v - \sin^2\gamma_v\right) + \right.$$
$$\left. + z_0^2\langle w_s\rangle^2 C_s^2\left(\tfrac{1}{4}\beta_v'^2\cos^2\gamma_v + \sin^2\gamma_v - \beta_v'\cos\gamma_v\sin\gamma_v\right)\right\}_{s=hC_s} +$$
$$+ \frac{55}{24\sqrt{3}}\,r_e\Lambda\gamma^5\bar{\Phi}^2\,\frac{1}{NC}\sum_1^{NM}{}_i\left\{\beta_v K^3\right\}_{s=s_i}\Delta s\,.$$

Averaging is performed on a number N of turns such that many betatron oscillations occur, while the syncrotron oscillations can be beglected.

The last term of (A.3) can be easily evaluated, since in the limit of $M\to\infty$, $\Delta s\to 0$ and

$$\text{(A.4)}\qquad \frac{1}{NC}\sum_1^{MN}{}_i\{\beta_v K^3\}_{s=s_i}\Delta s \to \frac{1}{NC}\int_0^{NC}\beta_v K^3\,\mathrm{d}s\,.$$

To evaluate the first term, note that, in the points $s=hC_s$, β_v and β_v' have always the same value, so that the sum can be written as a linear combination of terms of the form

$$\text{(A.5)}\qquad \{F(s)\}_{s=0}\cdot\frac{1}{NC}\sum_1^N{}_h\{f(\sin\gamma_v,\ \cos\gamma_v)\}_{s=hC_s}C_s\,.$$

If the betatron frequency is not an integer multiple of $2\pi/C_s$, the phase γ_v takes on all its possible values in the point $s=hC_s$, *i.e.* $\gamma_v(s+hC_s)$ for fixed s and variable h can be identified with the function $\gamma_v(s)$ so that

$$\text{(A.6)}\qquad \frac{1}{NC}\sum_1^N{}_h\{f(\sin\gamma_v,\ \cos\gamma_v)\}_{s=hC_s}C_s = \frac{1}{NC}\int_0^{NC}f(\sin\gamma_v,\ \cos\gamma_v)\,\mathrm{d}s\,.$$

On the other hand, it is always possible to choose N in such a way that the integral in (A.6) becomes almost equal to the integral over a certain number of betatron periods. It follows, for instance, that

$$\int_0^{NC}\sin\gamma_v\cos\gamma_v\,\mathrm{d}s = 0\,.$$

and so on.

Using (A.4), (A.6) it is now easy to evaluate (A.3) and to obtain (37) and (38).
In the case of the radial betatron oscillations and synchrotron oscillations, in addition to terms like (A.4), (A.5) one has also terms of the form

$$\frac{1}{NC}\sum_{1}^{NM}{}_i \{\{F(s)f(\sin\gamma,\ \cos\gamma)\}_{s=s_i}\Delta s\ . \tag{A.7}$$

$F(s)$ is a periodic function with period L and, if n is the number of periods of the machine, $C_s = nL$.
Introducing $m = M/n$, (A.7) becomes

$$\frac{1}{NC}\sum_{1}^{nN}{}_h\sum_{1}^{m}{}_j \{F(s)f(\sin\gamma,\ \cos\gamma)\}_{s=s_{j,h}}\Delta s\ ,$$

where a correspondence $i \to (h, j)$ has been established.
Now let us assume that approximately $s_{j,h} \simeq s_{j,1} + hL$; $F(s_{j,h}) \simeq F(s_{j,1})$; then it follows that

$$\frac{1}{NC}\sum_{1}^{NM}{}_i \{F(s)f(\sin\gamma,\ \cos\gamma)\}_{s=s_{j,h}}\Delta s = \tag{A.8}$$

$$= \frac{1}{NC}\sum_{1}^{nN}{}_h\sum_{1}^{m}{}_j \{F(s)f(\sin\gamma,\ \cos\gamma)\}_{s=s_{j,h}}\Delta s =$$

$$= \frac{1}{NCL}\sum_{1}^{m}{}_j \Delta s F(s_{j,1})\sum_{1}^{nN}{}_h \{f(\sin\gamma,\ \cos\gamma)\}_{s=s_{j,h}}L =$$

$$= \langle F(s)\rangle \frac{1}{NC}\int_0^{NC} f(\sin\gamma,\ \cos\gamma)\,\mathrm{d}s\ ,$$

It is then easy to get (43), (46).